

Continuous error correction

BY JUAN PABLO PAZ^{1,3} AND WOJCIECH HUBERT ZUREK^{2,3}

¹ *Departamento de Física J.J. Giambiagi, FCEN, UBA, Pabellón 1, Ciudad Universitaria, 1428 Buenos Aires, Argentina*

² *Theoretical Astrophysics, MSB288, Los Alamos National Laboratory, Los Alamos, NM 87545, USA*

³ *Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106-4030*

We propose a new approach to study the evolution of a quantum state that is encoded in a system which is continuously subject to the operations required to implement a quantum error correcting code. In the limit of *continuous error correction* we introduce a Markovian master equation that includes the effects of: a) Hamiltonian evolution, b) errors caused by the interaction with an environment and c) error-correcting operations. The master equation is formally presented for all stabilizer codes and its solution is analyzed for the simplest such code.

1. Introduction

In the last two years there has been enormous progress in the development of quantum error correction codes. The basic goal of these techniques is to protect quantum information against the degrading effect of uncontrolled interactions with external agents. To achieve this goal, being able to partially restore the unknown quantum state, one should perform a sequence of operations on the system used to represent the information. This active process, a quantum error correcting code, involves the following basic ingredients: First, one should encode the information (initially contained in the quantum state of k qubits) into an entangled quantum state of n carriers ($n > k$). After the encoding, the original state is protected from certain errors provided one acts on the system applying a so-called recovery operator. The simplest (but not the only) way to describe this operation is to decompose it in the following steps: one should first perform a measurement on $n - k$ qubits, whose result reveals a syndrome. This measurement is followed by a unitary operation on the remaining k qubits, whose nature depends upon the syndrome. Finally, the operation is completed by resetting the $n - k$ measured qubits to a reference (ground) state. The short history of quantum error correcting techniques is quite remarkable: after the first pioneering papers by P. Shor and A. Steane (Shor, 1995; Steane 1996*a,b*), several papers helped in clarifying the nature of quantum codes (Laflamme *et al.*, 1996; Bennett *et al.*, 1996). Very soon, the mathematical structure behind these simple physical ideas was revealed (Knill & Laflamme, 1997) and the, rather general, theory of stabilizer codes was fully developed (Gottesman, 1996*a*, 1997; Calderbank *et al.*, 1996). More recently, it was shown that the use of concatenated quantum error

correcting codes (Knill & Laflamme, 1996) together with fault tolerant techniques for quantum computation (Shor, 1997) may allow arbitrarily long computation provided the accuracy per operation is below certain computational threshold (Knill, Laflamme & Zurek, 1997; Preskill, 1997).

In this paper we propose a new tool that could be used to study the evolution of an encoded quantum state. The motivation of our method lies on the interesting analogy between the active mechanism of quantum error correction and the process of cooling a system pumping energy away from it by driving certain internal transitions. Thus, we propose to use a master equation (inspired in the quantum-optical master equation) to describe the limit in which a system formed by n -carrier qubits is continuously being affected by the active process of error correction while, at the same time, is suffering the continuous interaction with an environment and, eventually, is evolving under a specified Hamiltonian. The limit of continuous error correction is the one in which one assumes that all operation involved in the code (encoding, applying the recovery operator, refreshing the syndrome, etc) can be considered to be instantaneous for all practical purposes.

Our purpose here is twofold. On the one hand we will introduce the general master equation for a rather large class of error correcting codes (stabilizer codes). On the other hand, as we will show that this master equation can be explicitly solved in some simple but relevant cases, we will discuss important features of the solution. We find it rather interesting that quantum error correcting codes admit a description in terms of a continuous differential equation. But it is certainly most remarkable that this equation not only can be formally written in terms of some abstract operators but can also be explicitly solved in some interesting cases. The rest of the paper is organized as follows: In Section 2 we present a very brief introduction to the theory of stabilizer error correcting codes. The ideas discussed in this section enable us to introduce the notion of “corrective quantum jumps”. In Section 3, this notion is used to motivate a master equation describing the evolution of a quantum system which is continuously being affected by errors and is subject to a continuous sequence of corrective operations. We analyze and solve the master equation for the simplest example of a stabilizer code: the $n = 3$ qubit code protecting $k = 1$ qubit of information against phase errors affecting any one of the three carriers. In this case we present the explicit solution of the master equation computing the fidelity from the density matrix of the system. We discuss the various time scales that affect the evolution of the encoded state in the collective Bloch sphere.

2. Stabilizer error correcting codes.

We will consider a quantum error correcting code that protects k qubits by encoding them into n carriers. The code space \mathcal{H}_k is a 2^k dimensional subspace of the Hilbert space of the n carriers. \mathcal{H}_n is a tensor product of n 2-dimensional factors and has a natural basis whose elements are product states of the individual carriers. This “physical basis” can be formed with the common eigenstates of the operators $\{Z_1, \dots, Z_n\}$ (X_i, Y_i and Z_i denote the Pauli matrices for the i -th qubit). We will label states of this basis not by the eigenvalues of the corresponding operators (which are ± 1) but by the eigenvalues of the projectors onto the -1 subspace (which are 0 or 1): Thus, the label $z_j = 0$ ($z_j = 1$) corresponds to a $+1$

(-1) eigenvalue of the operator Z_j . Furthermore, we order the n carriers in such a way that the last k qubits are the ones whose state we encode while the first $n - k$ are the ancillary carriers. Therefore, states of the physical basis are of the form $|s, z\rangle_P = |s\rangle_P \otimes |z\rangle_P$ (where the strings $s = (s_1, \dots, s_{n-k})$, $z = (z_1, \dots, z_k)$ store the corresponding eigenvalues and the subscript P is used to identify states of the physical basis).

An error correcting code is a mapping from the physical product states $|0\rangle_P \otimes |\Psi\rangle_P$ onto the code space \mathcal{H}_k , which is formed by entangled states of the n carriers. A rather general class of codes can be described in terms of their stabilizer group (see Gottesman, 1997). The stabilizer of the code is an Abelian group formed by all operators which are tensor products of Pauli matrices and have \mathcal{H}_k as an eigenspace with eigenvalue $+1$. Every element of the stabilizer, which is a finite group with 2^{n-k} elements, can be obtained by appropriately multiplying $n - k$ generators, which will be denoted as M_1, \dots, M_{n-k} . The elements of the stabilizer are completely degenerate in the code space \mathcal{H}_k (since all states in \mathcal{H}_k are eigenstates with eigenvalue $+1$ of all M_j). To define a basis in the code space we choose k operators L_1, \dots, L_k which being tensor products of Pauli matrices commute with all elements of the stabilizer. These operators $L_{j'}$, $j' = 1, \dots, k$ are the “logical pointers” since they define the directions in \mathcal{H}_k associated with the logical states $|0\rangle_L, \dots, |2^k - 1\rangle_L$ (logical pointers belong to the group of operators which commute with the stabilizer, known as the normalizer).

The $n - k$ generators of the stabilizer together with the k logical pointers are a Complete Set of Commuting Operators (CSCO) whose common eigenstates form a complete basis of the Hilbert space \mathcal{H}_n . Elements of this “logical basis”, labeled by n quantum numbers, are denoted as $|m, l\rangle_L$, where the bit strings $m = (m_1, \dots, m_{n-k})$, $l = (l_1, \dots, l_k)$ identify the corresponding eigenvalues and the subscript L refers to logical states. The CSCO formed by the generators of the stabilizer and the logical pointers defines a prescription for decomposing the original Hilbert space of the n -carriers into a tensor product of a 2^k -dimensional logical space \mathcal{A} and a 2^{n-k} -dimensional syndrome space \mathcal{S} . In fact, elements of the logical basis (which are entangled states of the n -carriers) are tensor products of states belonging to \mathcal{A} and \mathcal{S} : $|m, l\rangle_L = |m\rangle_L \otimes |l\rangle_L$. Encoded states, which belong to \mathcal{H}_k , are also product states of the form $|\Psi\rangle = |0\rangle_L \otimes \sum_l c_l |l\rangle_L$.

The code protects quantum states against any error E_a whose action on states of the logical basis is to change the logical syndrome and, eventually, rotate the logical state in \mathcal{A} in a syndrome dependent way:

$$E_a |m\rangle_L \otimes |l\rangle_L = e^{i\phi_{ma}} |m + c_a\rangle_L \otimes U_a |l\rangle_L. \quad (2.1)$$

Here U_a is a unitary operator acting on the collective logical space \mathcal{A} and ϕ_{ma} is a phase that may depend on the syndrome and the error. The error E_a changes the syndrome from m to $m + c_a$ where c_a is the bit string storing the commutation pattern between the error and the generators of the stabilizer (the j -th bit of this string is one if the error anti-commutes with M_j and is zero otherwise). The reason for this is that when acting on a logical state, the error E_a changes the eigenvalue of the operator M_j only if $\{M_j, E_a\} = 0$. The label a used to identify errors is arbitrary and, for the case of non-degenerate codes (which is the only one we will consider here) it is always possible to label errors E_a using simply the commutation pattern c_a (i.e., we can choose $a = c_a$).

To correct against the action of any of the errors E_a (or against any linear

superposition of them) one can first detect the error by measuring the collective syndrome (i.e., measuring the observables M_j , $j = 1, \dots, n - k$) and later recover from the error by applying the corresponding operator U_a^\dagger . This detection–recovery process can be conveniently described as a quantum operation defined by the following mapping from the erroneous density matrix ρ_{in} into the corrected one ρ_{out} :

$$\rho_{out} = \sum_{m=0}^N R_m \rho_{in} R_m^\dagger \quad (2.2)$$

where the sum runs over all syndromes ($N = 2^{n-k} - 1$) and the recovery operator is

$$R_m = |0\rangle_L {}_L\langle m| \otimes U_m^\dagger. \quad (2.3)$$

By construction, these operators satisfy the identity $\sum_{m=0}^N R_m^\dagger R_m = I$.

As our description of error detection–recovery process is entirely formulated in the logical basis it does not involve any reference to encoding–decoding operations which can be simply defined as a change of basis: The encoding operator C is a unitary operator mapping the physical basis, formed by product states of the n carriers, onto the logical basis, formed by entangled states. Accordingly, C transforms the operators Z_i (whose eigenvalues define states in the physical basis) into the operators M_j , $L_{j'}$ (that label states in the logical basis). Thus, the encoding operator C is such that $Z_j = C^\dagger M_j C$, $j = 1, \dots, n - k$ and $Z_{n-k+j'} = C^\dagger L_{j'} C$, $j' = 1, \dots, k$. Taking this into account the action of the operator R_m can be described, in the physical basis, as the following sequence of operations: i) decode the state, ii) measure the syndrome in the physical basis by measuring Z_j in the first $(n - k)$ carriers, iii) If the result of the measurement is the string s , apply the syndrome dependent recovery operator U_s^\dagger resetting the syndrome back to zero, iv) encode the resulting state.

Finding a stabilizer code correcting a given set of errors is a rather hard task which involves designing generators having appropriate commutation patterns with the errors. Once the generators are found and the logical pointers are chosen an encoding–decoding operator can be constructed (strategies for designing encoding–decoding circuits from the stabilizer are known; see Gottesman, 1997). The recovery operators depend on the encoding–decoding strategy and can be explicitly found from the encoding circuit by running errors through it.

In the next Section we will illustrate our model for continuous error correction using the simplest stabilizer code: the one encoding $k = 1$ qubit using $n = 3$ carriers correcting against phase errors in any of the carriers. The basic errors are $E_1 = Z_1$, $E_2 = Z_2$ and $E_3 = Z_3$. The stabilizer of the code can be chosen to be generated by: $M_1 = X_1 X_3$ and $M_2 = X_2 X_3$. The commutation pattern associated with each error is $c_1 = 01$ (because the error Z_1 commutes with M_1 and anti-commutes with M_2), $c_2 = 10$, $c_3 = 11$ (note that we are labeling errors with the commutation pattern). An encoding circuit is shown in Figure 1. The operator C has the following simple properties:

$$C^\dagger Z_1 C = X_1, \quad C^\dagger Z_2 C = X_2, \quad C^\dagger Z_3 C = X_1 X_2 Z_3. \quad (2.4)$$

These properties entirely determine the action of the errors Z_i in the logical basis. For example, the last identity implies that $E_3 |m\rangle_L |l\rangle_L = (-1)^l |m+c_3\rangle_L |l\rangle_L$. Thus,

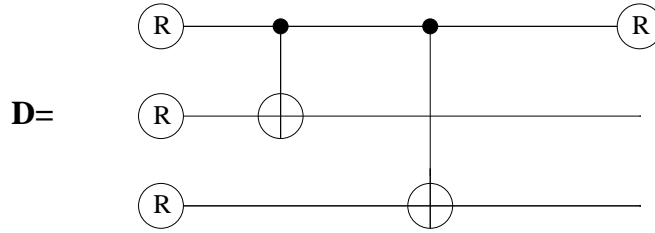


Figure 1. Encoding circuit for the three qubit code protecting against phase errors in the physical basis. Qubits are labeled from bottom to top. The third qubit is the one to be encoded. When run from left to right the circuit defines the decoding operator D . The encoding operator corresponds to the same circuit run backwards.

the error E_3 not only changes the syndrome but also modifies the logical state by adding a phase. This means that the recovery operator for this error is $U_3 = Z$. Analogously, we can find how the other errors act on the logical basis showing that $U_1 = U_2 = I$.

3. Master equation with corrective quantum jumps.

Quantum error correction is an active process: for the density matrix to evolve according to the rule (2.2), a sequence of physical operations must be carried out on the system (measuring the syndrome, refreshing it and applying the syndrome dependent recovery operator). Let us suppose that we are able to perform detection–recovery operations arbitrarily fast (i.e., the time required for these operations is negligible with respect to any other time scale in the system) and that the corrective protocol is repeatedly applied many times. In the limit of continuous error correction we will consider that the time required to perform the corrective operations goes to zero while the number of corrective events goes to infinity. In this limit the evolution of the density matrix is generated by the continuous application of the mapping (2.2).

In this paper we suggest that this rather idealized scenario, which might be useful to analyze the optimal protection achievable with a given code, could be described using a simple master equation. To motivate the master equation one could notice that the mapping from the erroneous to the corrected density matrix (given by (2.2)) is naturally described in the language of quantum jumps: In fact, the operators R_m define “corrective jumps” which reset the logical syndrome to $m = 0$ while acting as U_m^\dagger in the logical space \mathcal{A} . Thus, to describe the continuous limit we propose to use a Markovian master equation (analogous to the quantum optical master equation, see for example Walls & Milburn, 1994) having $N + 1$ decay channels associated with the R_m –quantum jump operators:

$$\dot{\rho} = L_0\rho + \sum_m \gamma_m \left(R_m\rho R_m^\dagger - \frac{1}{2}R_m^\dagger R_m\rho - \frac{1}{2}\rho R_m^\dagger R_m \right). \quad (3.1)$$

Here, the constants γ_m measure the strength of the corrective jumps and $L_0\rho$ includes all other sources of evolution (produced by a Hamiltonian H_0 or caused by the errors E_a). To write this non–corrective piece of the master equation we will

consider a system with a Hamiltonian H_0 that is interacting with an environment. We assume that this interaction produces a term in the master equation which, in the Markovian limit, has the same form than (3.1) with N decay channels E_a :

$$L_0\rho = -i[H_0, \rho] + \sum_{a=1}^N \gamma'_a \left(E_a \rho E_a^\dagger - \frac{1}{2} E_a^\dagger E_a \rho - \frac{1}{2} \rho E_a^\dagger E_a \right). \quad (3.2)$$

The complete master equation can be written in a rather compact form if we assume that all corrective jumps have equal strength ($\gamma_m = \gamma$) and that the same is true for the erroneous jumps ($\gamma'_a = \gamma'$). In this case, and if the erroneous jump operator are unitary, the complete master equation reads:

$$\dot{\rho} = -i[H_0, \rho] - (\gamma + N\gamma')\rho + \gamma \sum_{m=0}^N R_m \rho R_m^\dagger + \gamma' \sum_{a=1}^N E_a \rho E_a^\dagger. \quad (3.3)$$

To solve this equation it is convenient to remember that, as the Hilbert space \mathcal{H}_n is a tensor product of a collective syndrome space \mathcal{S} and a collective logical space \mathcal{A} , we can always write the density matrix ρ as

$$\rho = \sum_{m,m'=0}^N |m\rangle_{LL} \langle m'| \otimes \rho_{mm'}, \quad (3.4)$$

where the operators $\rho_{mm'}$ act on the 2^k -dimensional space \mathcal{A} . For ρ to be hermitian and normalized, the following identities must be satisfied: i) $\rho_{mm'}^\dagger = \rho_{m'm}$ and ii) $\sum_m \text{Tr}(\rho_{mm}) = 1$. Using this notation one can rewrite both the corrective and the error-generating terms in (3.3) as

$$\sum_{m=0}^N R_m \rho R_m^\dagger = |0\rangle_{LL} \langle 0| \otimes \sum_{m=0}^N U_m^\dagger \rho_{mm} U_m, \quad (3.5)$$

$$\sum_{a=1}^N E_a \rho E_a^\dagger = \sum_{a=1}^N \sum_{mm'=0}^N |m+a\rangle_{LL} \langle m'+a| \otimes e^{i(\phi_{ma} - \phi_{m'a})} U_a \rho_{mm'} U_a^\dagger, \quad (3.6)$$

where errors are labeled using the commutation pattern c_a (i.e. we choose to take $a = c_a$). These equations show that neither the erroneous nor the corrective quantum jumps couple diagonal operators ($m = m'$) with off diagonal ones ($m \neq m'$). Using them, equations for diagonal operators $\rho_m = \rho_{mm}$ can be easily found. To explicitly write down the equations for ρ_m we will assume that the Hamiltonian H_0 does not connect states with different syndromes. For this type of Hamiltonian (which are the ones required for a fault tolerant computation) we have $H_0 = \sum_{m=0}^N |m\rangle \langle m| \otimes h_m$, where h_m is a Hamiltonian acting in the logical subspace \mathcal{A} . Using this, the master equation which includes the effects of Hamiltonian evolution, errors and error correction, reads:

$$\begin{aligned} \dot{\rho}_m = & -i[h_m, \rho_m] - (\gamma(1 - \delta_{m0}) + N\gamma')\rho_m + \\ & + \sum_{a=1}^N \left(\delta_{m0} \gamma U_a^\dagger \rho_a U_a + \gamma' U_a \rho_{m+a} U_a^\dagger \right). \end{aligned} \quad (3.7)$$

As we can see from (3.7), all operators tend to be damped by the corrective term

except ρ_0 . On the other hand, errors tend to induce transitions between different syndromes at a rate proportional to γ' . From these equations we can first compute the probability for detecting the m -th syndrome, which is nothing but the trace of ρ_m : defining $p_m = \text{Tr}(\rho_m)$ and taking the trace in (3.7) we find:

$$\dot{p}_m = -(\gamma(1 - \delta_{m0}) + N\gamma')p_m + \sum_{a=1}^N (\delta_{m0}\gamma p_a + \gamma' p_{m+a}). \quad (3.8)$$

Solving this system, we obtain the probability for the correct syndrome

$$p_0(t) = \frac{\gamma + \gamma'}{\gamma + (N+1)\gamma'} + \left(p_0(0) - \frac{\gamma + \gamma'}{\gamma + (N+1)\gamma'} \right) \exp(-\lambda t), \quad (3.9)$$

where the decay rate is $\lambda = \gamma + (N+1)\gamma'$. This rather simple formula, valid for all stabilizer codes in the limit of continuous correction, shows that the probability for detecting the correct syndrome tends to be of order unity if: i) the corrective coupling constant dominates over γ' ($\gamma \gg \gamma'$ defines the strong correction limit) and ii) times are large as compared to $t_c = 1/\gamma$.

To solve the master equation finding the operators ρ_m we will be more specific restricting ourselves to consider codes protecting $k = 1$ qubit. In this case the operator ρ_m acts on a 2-dimensional Hilbert space and can be written as a linear combination of Pauli matrices:

$$\rho_m = \frac{1}{2} (p_m \hat{1} + \vec{r}_m \hat{\sigma}). \quad (3.10)$$

where $\hat{\sigma}$ is the vector of Pauli matrices and the polarization vector \vec{r}_m is the expectation value of $\hat{\sigma}$ in the state ρ_m (note that ρ_m is not normalized since $p_m = \text{Tr}(\rho_m)$ is the probability for detecting the m -th syndrome). The master equation (3.7) translates into a linear system of equations for the polarization vectors \vec{r}_m . Thus, the evolution of the quantum state is determined by the dynamics of the vectors \vec{r}_m , each one of which lives in its own Bloch sphere. To obtain equations for \vec{r}_m we assume that the Hamiltonian h_m is proportional to a Pauli matrix (which induces a rotation of \vec{r}_m on the Bloch sphere). Thus, if $h_m = \Omega \sigma_k / 2$ we find

$$\dot{\vec{r}}_m = \vec{\Omega} \times \vec{r}_m - (\gamma(1 - \delta_{m0}) + N\gamma')\vec{r}_m + \sum_{a=1}^N \Lambda_a (\gamma \delta_{m0} \vec{r}_a + \gamma' \vec{r}_{m+a}). \quad (3.11)$$

Here, $\vec{\Omega} = \Omega \mathbf{k}$ and the matrix elements of the 3×3 matrices Λ_a are:

$$(\Lambda_a)_{ij} = \frac{1}{2} \delta_{ij} \text{Tr}(\hat{\sigma}_j U_a)^2. \quad (3.12)$$

It is worth noting that, as for stabilizer codes the operators U_a are Pauli matrices, the only possible values of the diagonal elements $\Lambda_{a,ii}$ are ± 1 . The equations for \vec{r}_m can be analytically solved in some simple cases. Here, we consider only the code protecting $k = 1$ qubit against phase errors affecting any one of the $n = 3$ carriers. In this case we can use the results given in the previous section to show that the matrices Λ_1 and Λ_2 are equal to the identity while $\Lambda_3 = \text{diag}(-1, -1, +1)$. Using this, we can obtain the equation for the polarization vector \vec{r}_0 (for convenience we avoid writing the subscript 0 and denote $\vec{r} = \vec{r}_0$). It turns out that \vec{r} is

coupled through (3.11) with two other vectors: $\vec{r}_I = \vec{r}_1 + \vec{r}_2$ (the sum of the two polarization vectors corresponding to syndromes for which the recovery operator is $U = I$) and $\vec{r}_Z = \vec{r}_3$ (the polarization vector corresponding to the syndrome for which the recovery operator is $U = Z$). Thus, one can show that (3.11) reduces to:

$$\dot{\vec{r}} = \vec{\Omega} \times \vec{r} - 3\gamma'\vec{r} + (\gamma + \gamma')(\vec{r}_I + \vec{r}_Z), \quad (3.13)$$

$$\dot{\vec{r}}_I = \vec{\Omega} \times \vec{r}_I - (\gamma + 3\gamma')\vec{r}_I + \gamma'(2\vec{r} + 2\vec{r}_Z + \Lambda_3\vec{r}_I), \quad (3.14)$$

$$\dot{\vec{r}}_Z = \vec{\Omega} \times \vec{r}_Z - (\gamma + 3\gamma')\vec{r}_Z + \gamma'(\vec{r}_I + \Lambda_3\vec{r}). \quad (3.15)$$

where $\vec{\Omega} = (\Omega, 0, 0)$. Details of the solution of this system will be analyzed elsewhere (Paz & Zurek, 1997). Here we will discuss the simplest case where an exact solution is possible. In fact, when $\Omega = 0$ (free evolution), the above system reduces to:

$$\dot{\vec{r}} = -3\gamma'\vec{r} + (\gamma + \gamma')\vec{v} \quad (3.16)$$

$$\dot{\vec{v}} = -(\gamma + 3\gamma')\vec{v} + 3\gamma'\vec{r} + 2\gamma'\Lambda_3\vec{v} \quad (3.17)$$

where the vector \vec{v} is defined as $\vec{v} = \sum_{a=1}^N \Lambda_a \vec{r}_a = \vec{r}_I + \vec{r}_Z$. Solving this system we can find the three components of the polarization vector $\vec{r} = (x, y, z)$. If the initial state belongs to the $m = 0$ syndrome subspace (i.e., $\rho(0) = |0\rangle\langle 0| \otimes \rho_0(0)$) we find:

$$z(t) = z_0 p_0(t) = z_0 \left(\frac{\gamma + \gamma'}{\gamma + 4\gamma'} + \left(1 - \frac{\gamma + \gamma'}{\gamma + 4\gamma'} \right) \exp(-\lambda t) \right), \quad (3.18)$$

$$y(t) = y_0 \frac{1}{\lambda_- - \lambda_+} ((3\gamma' - \lambda_+) \exp(-\lambda_- t) + (\lambda_- - 3\gamma') \exp(-\lambda_+ t)), \quad (3.19)$$

where the decay rates are $\lambda_{\pm} = 4\gamma' + \gamma/2 \mp \sqrt{4\gamma'^2 + 4\gamma\gamma' + \gamma^2/4}$ and $x(t)$ obeys an equation which is identical to that of $y(t)$.

Before discussing these results it is worth introducing a measure of the effectiveness of the error correcting process. For this we will use the state fidelity, which can be defined as

$$F = \text{Tr} \left(\rho_{id}(t) \sum_m R_m \rho(t) R_m \right), \quad (3.20)$$

where $\rho(t)$ is the solution of the full master equation and ρ_{id} is the ideal density matrix obtained by unitarily evolving the initial state using the appropriate Hamiltonian. For the simplest case we are considering here ($H_0 = 0$) the fidelity is determined by the scalar product between the initial state $\vec{r}(0)$ and the vector $(\vec{r} + \vec{v})(t)$. Thus, a simple calculation shows that $F = \frac{1}{2} \left(1 + \frac{1}{2} \vec{r}(0) \cdot (\vec{r} + \vec{v})(t) \right)$, which, using the above results, can be written as

$$F = \frac{1}{2} \left(\left(1 + z_0^2 \right) + \left(x_0^2 + y_0^2 \right) \frac{1}{\lambda_- - \lambda_+} (\lambda_- \exp(-\lambda_+ t) - \lambda_+ \exp(-\lambda_- t)) \right). \quad (3.21)$$

A first comment on the results is that there are some states for which the fidelity is always maximal. In fact, if the initial state is completely polarized along the z -axis, the code will always correct with ideal fidelity. The reason for this has

nothing to do with continuous error correction but is a simple consequence of the fact that for such states all the errors Z_i act trivially on the collective logical space \mathcal{A} . Only states which have non vanishing components on the (x, y) -plane are nontrivially affected by the errors we are considering here.

It is useful to analyze our results in various limiting situations. For the simplest case with no continuous correction ($\gamma = 0$), the fidelity is

$$F = \frac{1}{2} \left((1 + z_0^2) + (x_0^2 + y_0^2) \frac{1}{2} (3 \exp(-2\gamma' t) - \exp(-6\gamma' t)) \right). \quad (3.22)$$

This expression shows that, without continuous correction, the fidelity decays at a rate fixed by γ' (the initial decay is quadratic because the fidelity defined in (3.20) includes a final correction event at time t , which is sufficient to assure that for short times $F \approx 1 - 6\gamma'^2 t^2$). The effect of continuous error correction is clearly seen when comparing the above expression with the one arising in the limit of strong correction ($\gamma \gg \gamma'$) where, to first order in $\epsilon = \gamma'/\gamma$, we can approximate $\lambda_- \approx \gamma(1 + 8\epsilon + O(\epsilon^2))$ and $\lambda_+ \approx 12\gamma\epsilon^2$. In this case we find:

$$z(t) = z_0 (1 - 3\epsilon + 3\epsilon \exp(-\gamma t)), \quad (3.23)$$

$$y(t) = y_0 \left((1 - 3\epsilon) \exp(-12\epsilon^2 \gamma t) + 3\epsilon \exp(-\gamma t) \right), \quad (3.24)$$

$$F = \frac{1}{2} \left[(1 + z_0^2) + (x_0^2 + y_0^2) \times \right. \\ \left. \times \left((1 + 12\epsilon^2) \exp(-12\epsilon^2 \gamma t) - 12\epsilon^2 \exp(-\gamma t) \right) \right] \quad (3.25)$$

Two different time scales appear in the strong correction limit: On the one hand, the short time scale $t_c = 1/\gamma$ (after which the last terms in (3.23–3.25) are exponentially suppressed) is associated with the corrective jumps. Over this time scale the normalized state $\tilde{\rho} = \frac{1}{2} (\hat{I} + \vec{r} \hat{\sigma}/p_0)$ lies on the surface of the Bloch sphere and is effectively protected from errors. On the other hand, on the much longer time $t_2 = t_c/12\epsilon^2$ there is an overall exponential damping of the (x, y) -components of the Bloch vector which moves the normalized state $\tilde{\rho}$ towards the interior of the Bloch sphere. On this time scale the error correcting code ceases to be efficient: fidelity decreases and purity of the quantum state is lost.

Finally, we can briefly discuss the case where H_0 is nontrivial ($\Omega \neq 0$). For the three qubit code a fault tolerant rotation of the encoded state can be easily achieved by using the Hamiltonian $H_0 = \Omega X_3/2$. Thus, as the encoding operator is such that $CX_3C^\dagger = X_3$, applying $H_0 = \Omega X_3$ during an appropriate time interval (to make an exact π -pulse) we obtain an encoded bit-flip. The reason for this is that while the operator X_3 commutes with the generators M_j (and therefore does not change the syndrome) it anti-commutes with the logical pointer $L = Z_1 Z_2 Z_3$. Therefore, applying the operator X_3 is tantamount to an encoded X operator. With this Hamiltonian, the polarization vector \vec{r} evolves according to the system (3.13–3.15) which can be numerically solved. As the structure of the equation is similar to the one found in the free case one expects the results for the fidelity to be essentially the same we described above. The only relevant exception is that the initial state with $z_0^2 = 1$ no longer produces ideal fidelity. In Figure 2 we show the time dependence fidelity for such initial state and display the (y, z) components of the polarization vector \vec{r} .

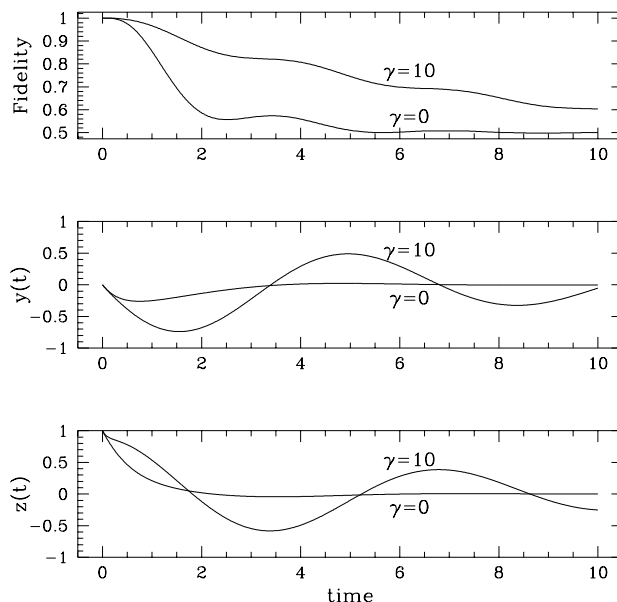


Figure 2. Fidelity and non vanishing components of the polarization vector with and without continuous error correction. The initial state is polarized along the \mathbf{z} axis and the Hamiltonian is $H_0 = \sigma_x/2$ (the strength of the erroneous jumps is also set to unity).

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